Temperature Profile in Lake Bottom Sediments: An Analytical Self-Similar Solution

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Abstract

The vertical temperature structure of bottom sediments in lakes is discussed. Observations indicate that the temperature profile in the bottom sediments has the form of a thermal wave. The wave starts at the water-sediment interface and propagates downward as the lake water and the bottom sediments are heated during spring and summer. When heating ceases and cooling sets in, a new wave starts at the water-sediment interface and propagates downward as the lake water and the sediments are cooled over autumn and winter. Although the temperature in the thermally active layer of bottom sediments varies considerably with time, a characteristic *shape* of the temperature-depth curve remains approximately the same. Based on this observational evidence, a two-layer parametric representation of the temperature profile in lake bottom sediments was proposed. In this note, a theoretical explanation for the observed self-similarity of the temperature profile is offered. Assuming a travelling wave-type behaviour of the temperature profile in bottom sediments, an analytical solution to the heat transfer equation is found. This solution is compared with data from measurements in a number of lakes and in a laboratory tank.

1 Introduction

A distinctive feature of most shallow lakes, ponds and other natural and man-made reservoirs is a strong thermal interaction between the water body and the bottom sediments. A sizable portion of the heat received from the atmosphere during spring and summer is accumulated in the thermally active upper layer of bottom sediments. This heat is then returned back to the water column during autumn and winter, leading to a hysteresis-like behaviour of the seasonal temperature cycle of the water column-bottom sediment system.

A straightforward approach to describe the vertical temperature structure of bottom sediments is to use the equation of heat transfer with a priori knowledge of the thermal diffusivity of sediments (see Gu and Stefan 1990, Fang and Stefan 1996, 1998, and references therein). The major shortcoming of this straightforward approach is that the thermal diffusivity is strongly dependent on the composition of the sediments and on the amount of organic matter they contain and therefore it is rarely well known.

Golosov and Kreiman (1992) proposed an alternative way of describing the vertical temperature structure of bottom sediments. Their approach is based on a two-layer self-similar parametric representation of the evolving temperature profile in the sediments that is conceptually similar to a parametric representation of the temperature profile in the upper mixed layer and the seasonal thermocline in the ocean (Kitaigorodskii and Miropolsky 1970). Using empirical polynomial approximations of the temperature-depth curve, a computationally-efficient bulk model for calculating the temperature structure of bottom sediments and the heat flux through the water-sediment interface was developed (Golosov et al. 1998). The concept of self-similarity

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of the temperature profile in bottom sediments rests on empirical evidence and has received no theoretical explanation so far. In the present note, a plausible theoretical explanation for the observed self-similarity is offered in terms of the travelling wave-type solution to the heat transfer equation.

2 Background

Observations suggest (a summary of observational studies is given by Ryanzhin 1997) that the temperature profile in bottom sediments has the form of a thermal wave. Typical temperature profiles in the lake bottom sediments are illustrated schematically in Fig. 1. The wave starts at the water-sediment interface $z = D$ and propagates downward as the lake water and the bottom sediments are heated during spring and summer. When heating ceases and cooling sets in, a new thermal wave starts at $z = D$. It propagates downward as the lake water and the sediments are cooled during autumn and winter, thus closing the annual cycle. The layer $D \leq z \leq L$, where seasonal temperature changes take place, is the thermally active layer of bottom sediments. Below the level $z = L$ the temperature changes are negligibly small. Importantly, a characteristic *shape* of the temperature-depth curve remains approximately the same, although the temperature at different depths varies considerably with time. Motivated by this empirical evidence, a two-layer self-similar parametric representation of the temperature profile in the bottom sediments was proposed by Golosov and Kreiman (1992) and further developed by Golosov et al. (1998). The expression of Golosov et al. (1998) reads

$$\theta(z,t) = \begin{cases} \theta_D(t) + [\theta_H(t) - \theta_D(t)]\Theta(\zeta) & \text{at} \quad D \leq z \leq H(t) \\ \theta_H(t) + [\theta_L - \theta_H(t)]\vartheta(\xi) & \text{at} \quad H(t) \leq z \leq L. \end{cases} \tag{1}$$

Here, t is time, $\theta(z,t)$ is the temperature, θ_L is the (constant) temperature at the outer edge $z = L$ of the thermally active layer of the sediments, $\theta_D(t)$ is the temperature at the water-sediment interface, $\theta_H(t)$ is the temperature at the depth $H(t)$ penetrated by the wave, and $\Theta \equiv [\theta(z,t) - \theta_D(t)]/[\theta_H(t) - \theta_D(t)]$ and $\vartheta \equiv [\theta(z,t) - \theta_H(t)]/[\theta_L - \theta_H(t)]$ are dimensionless universal functions of dimensionless depths $\zeta \equiv [z - D]/[H(t) - D]$ and $\xi \equiv [z - H(t)]/[L - H(t)]$, respectively. These functions, often referred to as shape functions, are universal in the sense that they are independent of time, although the temperature and the depth penetrated by the wave are time dependent. Using empirical polynomial approximations of $\Theta(\zeta)$ and $\vartheta(\xi)$, Golosov et al. (1998) developed a simple procedure for calculating the heat flux through the water-sediment interface. Simulations of the seasonal cycle of temperature in the bottom sediments of several lakes using this procedure showed a satisfactory agreement with observations (Golosov et al. 1998, Kondratiev et al. 1998).

Notice that the temperature profile below the level $z = H$ penetrated by the thermal wave does not remain "frozen". Some temperature changes do occur at $z > H$ through molecular heat conductivity due to the non-linearity of the initial temperature profile. These changes are, however, very slow as compared to the temperature changes in the layer $D \leq z \leq H$. A model of the vertical temperature structure of bottom sediments, where $\partial\theta/\partial t = 0$ at $z > H$, i.e. the point ($z = H$, $\theta = \theta_H$) slides down the initial temperature profile, was considered by Golosov and Kreiman (1992). Here, no consideration is given to the temperature distribution at $z > H$. We focus attention on the layer $D \leq z \leq H$ where major temperature changes take place.

There is a close analogy between the self-similarity of the temperature profile in bottom sediments and the self-similarity of the temperature profile in the upper mixed layer and in the seasonal thermocline in the ocean and lakes. Using the mixed-layer temperature θ_s and its depth

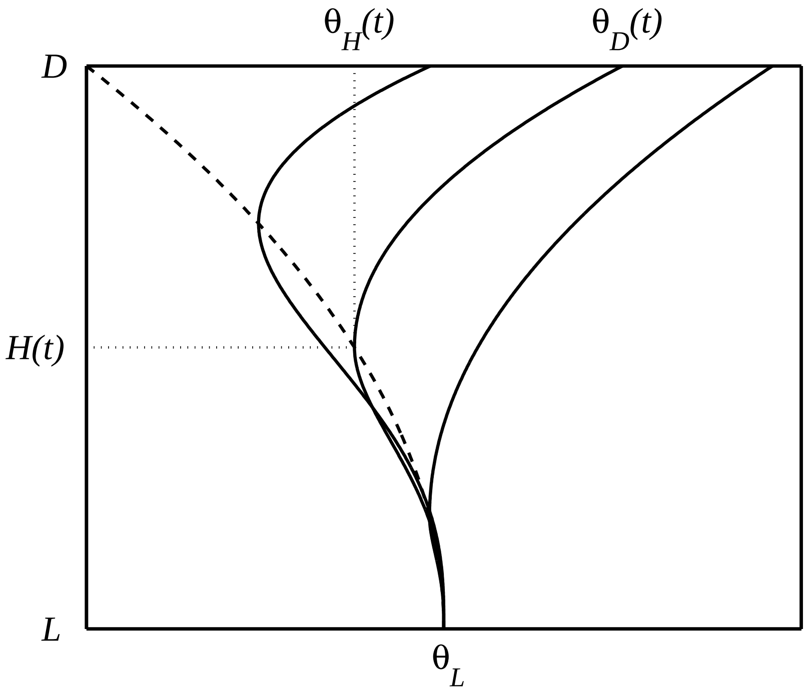

Figure 1: Schematic representation of the temperature profiles in bottom sediments during the period of heating. Dashed curve shows the initial temperature profile, i.e. the profile developed towards the end of the previous period of cooling. The profiles during the period of cooling are mirror images of the profiles during the period of heating.

h, the mixed layer concept can be expressed as $\theta(z,t)/\theta_s(t) = \Phi_{ML}[z/h(t)]$, where a dimensionless universal function Φ_{ML} is simply a constant equal to one. The concept of the temperature uniform mixed layer has been extensively used in geophysical fluid dynamics over several decades. Its natural extension is the concept of self-similarity of the temperature profile in the thermocline. It was put forward by Kitaigorodskii and Miropolsky (1970) to describe the vertical temperature structure of the oceanic seasonal thermocline. The essence of the concept is that the dimensionless temperature profile in the thermocline can be parameterized through a universal function of dimensionless depth, using the temperature difference $\Delta\theta$ across the thermocline and its thickness Δh as appropriate scales of temperature and depth, respectively. That is

$$\frac{\theta_s(t) - \theta(z,t)}{\Delta\theta(t)} = \Phi_T\left[\frac{z - h(t)}{\Delta h(t)}\right] \quad \text{at } \; h(t) \le z \le h(t) + \Delta h(t), \tag{2}$$

where a dimensionless function Φ_T, the shape function, is not merely a constant, as is the case with Φ_{ML} in the mixed layer, but a more sophisticated function of dimensionless depth. Notice that in small-to-medium depth reservoirs, the thermocline is pressed against the bottom. The abyssal quiescent layer is usually absent so that $\Delta\theta = \theta_s - \theta_D$ and $\Delta h = D - h$.

The concept of self-similarity of the temperature profile in the thermocline received support through observational (e.g. Filyushkin and Miropolsky 1981, Mälkki and Tamsalu 1985, Zilitinkevich 1991) and laboratory (e.g. Linden 1975, Wyatt 1978) studies. It proved to be a useful phenomenological alternative to a rigorous theory of turbulent heat transfer in strongly stable stratification characteristic of the thermocline. The concept has been successfully applied to model the oceanic seasonal thermocline (e.g. Kitaigorodskii and Miropolsky 1970, Arsenyev and Felzenbaum 1977, Filyushkin and Miropolsky 1981), the temperature inversion capping the atmospheric convectively mixed layer (e.g. Deardorff 1979, Fedorovich and Mironov 1995) and seasonal temperature changes in fresh-water lakes (e.g. Zilitinkevich 1991, Mironov et al. 1991).

A plausible theoretical explanation for the observed self-similarity of the temperature profile in the thermocline was provided in case of the mixed-layer deepening, $dh/dt > 0$ (Barenblatt 1978, Turner 1978, Shapiro 1980, Zilitinkevich et al. 1988, Zilitinkevich and Mironov 1992, Kirillin 2001). Introducing a vertical co-ordinate moving with the mixed layer-thermocline interface and assuming constant temperatures at the upper and lower boundaries of the thermocline, these authors considered a travelling wave-type solution to the heat transfer equation,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} K \frac{\partial \theta}{\partial z}, \tag{3}$$

where K is the temperature diffusivity that was either assumed to be constant, or was taken to be dependent on the temperature (density) stratification in the thermocline. The travelling wave-type analytical solutions to Eq. (3) proved to be fairly similar to empirical polynomial approximations of $(\theta_s - \theta)/\Delta\theta$ versus $(z - h)/\Delta h$ curve. We note that in case of the mixed-layer stationary state or retreat, $dh/dt \leq 0$, no theoretical explanation for the observed self-similarity of the temperature profile in the thermocline has been provided so far. The self-similarity at $dh/dt \leq 0$ should, therefore, be considered as purely phenomenological.

The thermal wave in the bottom sediments propagates downward both over the period of heating and over the period of cooling, so that $dH/dt > 0$. It is then tempting to find an analytical travelling wave-type solution to the heat transfer equation that would provide a theoretical explanation for the observed self-similarity of the temperature profile in bottom sediments. Such solution is presented in the next section.

3 Analytical Solution

Consider the simplest case of constant temperature diffusivity K of the bottom sediments. Introducing dimensionless variables $\Theta \equiv (\theta - \theta_D)/(\theta_H - \theta_D)$ and $\zeta \equiv (z - D)/(H - D)$, the equation of heat transfer (3) takes the form

$$\frac{d^2\Theta}{d\zeta^2} + E\zeta\frac{d\Theta}{d\zeta} - (\Pi_D - \Pi_H)\Theta = -\Pi_D, \tag{4}$$

where $E = K^{-1}(H - D)dH/dt$ is the dimensionless rate of propagation of the thermal wave, and $\Pi_D = K^{-1}(\theta_D - \theta_H)^{-1}(H - D)^2 d\theta_D/dt$ and $\Pi_H = K^{-1}(\theta_D - \theta_H)^{-1}(H - D)^2 d\theta_H/dt$ are the dimensionless time-rates-of-change of the temperature at the water-sediment interface $z = D$ and at the depth $z = H$ penetrated by the wave, respectively.

Taking E, Π_D and Π_H to be constant (restrictions these assumptions impose on the feasibility of the results are discussed in the next section), Eq. (4) subject to the boundary conditions $\Theta(0) = 0$ and $\Theta(1) = 1$ has an analytical solution in the form

$$\Theta = \frac{\Pi_D}{\Pi_D - \Pi_H} \times$$

$$\left(1 + \frac{\exp\left(-E\zeta^2/4\right)}{\zeta^{1/2}} \left\{ P \left[\frac{W_{p,1/4}(E/2)}{M_{p,1/4}(E/2)} M_{p,1/4}(E\zeta^2/2) - W_{p,1/4}(E\zeta^2/2) \right] - \frac{\Pi_H}{\Pi_D} \exp(E/4) \right\} \right), \tag{5}$$

where $P = 2^{1/4}\pi^{-1/2}E^{-1/4}\Gamma\left[(2E + \Pi_D - \Pi_H)/2E\right]$, $p = -\left[2(\Pi_D - \Pi_H) + E\right]/4E$, Γ is the Gamma function, and M and W are the Whittaker functions (Abramowitz and Stegun 1964, Chapter 13). The solution (5) describes a family of temperature-depth curves where the shape of the curve depends upon E, Π_D and Π_H.

4 Comparison with Data

In Fig. 2, the analytical solution (5) is compared with data from measurements in Lake Krasnoe, ca. 80 km north of St. Petersburg, Russia (Kuzmenko 1976, 1984), in Lake Velen, southern Sweden (Thanderz 1973) and in Lake Mendota, Wisconsin, USA (Birge et al. 1927), and with data from laboratory experiments (Golosov et al. 1998).

Measurements of temperature in the bottom sediments of Lake Krasnoe were taken during the period from 1972 to 1985 at a number of stations with the water depth ranging from 3 m to 10 m. Measurements were taken with a slow-response temperature probe mounted on the tip of a thin metal rod. The rod was plunged into the sediment to position the probe at various depths where readings were taken. The spacing between the depths of measurements was from 0.25 m to 0.5 m, with the uppermost reading taken at the water-sediment interface and the lowermost reading 3–4 m beneath the interface. The temperature profiles were recorded three times a month when the lake was free of ice and once a month during the period of ice cover. The profiles selected for comparison with analytical results were taken at stations with the water depth of 10 m and 3 m. The data shown in Fig. 2 were taken during 1976, 1983 and 1980 and are representative of anomalously cold, of anomalously warm, and of climatologically mean weather conditions, respectively. Measurements in Lake Velen were performed during February – September 1971 at the station with the water depth of 11.5 m; details are given in Thanderz (1973). The temperature profiles shown in Fig. 2 are taken from temperature-depth plots. Also presented in Fig. 2 are data from the pioneering paper of Birge et al. (1927). These authors were apparently the first to consider the temperature distribution in lake bottom sediments based on measurements in Lake Mendota. Measurements were performed during the period from 1916 to 1921 at a number of stations with the water depth ranging from 8 m to 23.5 m.

Laboratory data on the temperature distribution in bottom sediments are taken from Golosov et al. (1998). Experiments were performed in a rectangular tank 50 cm$\times$20 cm in cross-section, filled with 5 cm of water and 10 cm of sediment. Silt and sand were used as the working media. Heating of the sediment was accomplished by circulating warm water through the tank. The water temperature was controlled with a thermostat. Cooling of the sediment was achieved by adding ice to the system to keep the water temperature close to the freezing point. Temperature records were obtained with a thermistor chain. The chain has 7 thermistors with 1.5 cm spacing, the uppermost sensor being at the water-sediment interface. An experiment with the silt consisted of a period of heating over 1.75 hours, followed by a period of cooling over 1 hour. An experiment with the sand consisted of a period of heating over 1 hour. The temperature profiles were recorded at 5 to 10 minute intervals.

As we have mentioned in the previous section, the analytical solution (5) is conditioned by a constant dimensionless propagation rate of the thermal wave, $E = K^{-1}(H - D)dH/dt$, and constant time-rates-of-change of the temperatures at the water-sediment interface and at the depth penetrated by the wave, $\Pi_D = K^{-1}(\theta_D - \theta_H)^{-1}(H - D)^2 d\theta_D/dt$ and $\Pi_H = K^{-1}(\theta_D - \theta_H)^{-1}(H - D)^2 d\theta_H/dt$, respectively. In case E, Π_D and Π_H are not constant but vary slowly with time, the self-similarity of the temperature profile in bottom sediments is approximate. If these parameters undergo fast changes, no self-similarity is expected. The condition $E = const$ implies that the depth penetrated by the thermal wave grows as $H - D = (2EKt)^{1/2}$. This 1/2 power law is consistent with empirical and laboratory data (see Golosov et al. 1998), except at the initial stage and at the final stage of the development of the thermal wave. We therefore excluded from the analysis the temperature profiles with H close to D and H close to L. The conditions $\Pi_D = const$ and $\Pi_H = const$ imply that $\theta_D/\theta_* = \Pi_D(\Pi_D - \Pi_H)^{-1}\,(t/t_*)^{(\Pi_D - \Pi_H)/2E}$ and $\theta_H/\theta_* = \Pi_H(\Pi_D - \Pi_H)^{-1}\,(t/t_*)^{(\Pi_D - \Pi_H)/2E}$, where θ_* and t_* are the temperature and time scales. These power laws (that obey initial conditions $\theta_H = \theta_D = 0$ at $t = 0$) describe a broad spectrum of time dependencies of θ_D and θ_H and therefore impose rather mild restrictions on the feasibility of analytical results.

Following Golosov et al. (1998), the depth $z = H$ penetrated by the wave is determined from the measured temperature profiles as the level of zero vertical temperature gradient. As this is not the only conceivable way to determine H, some uncertainty may be introduced. It is likely to be small, however, considering that the temperature changes below the level of zero temperature gradient are small for the majority of profiles used for the analysis. The dimensionless parameters E, Π_D and Π_H are estimated from the measured temperature profiles, using finite-difference approximations of time derivatives of H, θ_D and θ_H. We find that the values $E = 1$, $\Pi_D = 2$ and $\Pi_H = 1$ and $E = 2$, $\Pi_D = 4$ and $\Pi_H = 2$ cover for the most part the range of conditions characteristic of the lakes and of the laboratory experiments considered in the present study. Theoretical curves plotted in Fig. 2 correspond to three different combinations of the dimensionless parameters: $E = 1$, $\Pi_D = 2$ and $\Pi_H = 1$; $E = 1.5$, $\Pi_D = 3$ and $\Pi_H = 1.5$; and $E = 2$, $\Pi_D = 4$ and $\Pi_H = 2$. As Fig. 2 suggests, the analytical results are in good agreement with observational and laboratory data. On the average, the curve that corresponds to $E = 1.5$, $\Pi_D = 3$ and $\Pi_H = 1.5$ gives the best fit to the data. The scatter of data is substantial, however.

Also shown in Fig. 2 is a phenomenological approximation of $\Theta(\zeta)$ proposed by Golosov et al. (1998), using a geometrical approach. These authors expressed Θ as a polynomial in ζ and invoked boundary conditions to specify the polynomial coefficients. The conditions $\Theta(0) = 0$ and $\Theta(1) = 1$ follow from the definition of Θ and ζ. The third condition is the zero temperature gradient at the depth penetrated by the thermal wave, that is $d\Theta(1)/d\zeta = 0$. The resulting expression is the second-order polynomial

$$\Theta = 2\zeta - \zeta^2, \tag{6}$$

which is the simplest polynomial that satisfies a minimum set of constraints. As seen from Fig. 2, the polynomial (6) lies in the region bounded by the upper and the lower analytical curves. By and large it slightly underestimates the data.

5 Conclusions

The vertical temperature profile in bottom sediments in lakes, ponds and other natural and man-made reservoirs was observed to be self-similar, having the form of a thermal wave that propagates from the water-sediment interface downward. The self-similarity means that, although the temperature in the thermally active upper layer of bottom sediments varies considerably with time, a characteristic shape of the temperature-depth curve remains approximately the same.

A theoretical explanation for the observed self-similarity of the temperature profile in bottom sediments is offered. Assuming a travelling wave-type behaviour of the temperature profile, an analytical solution to the heat transfer equation is found. This solution compares favourably with data from measurements in a number of lakes and with data from laboratory experiments. The analytical solution, Eq. (5), appears to be fairly similar to a phenomenological polynomial approximation of the temperature profile, Eq. (6), developed on the basis of empirical data.

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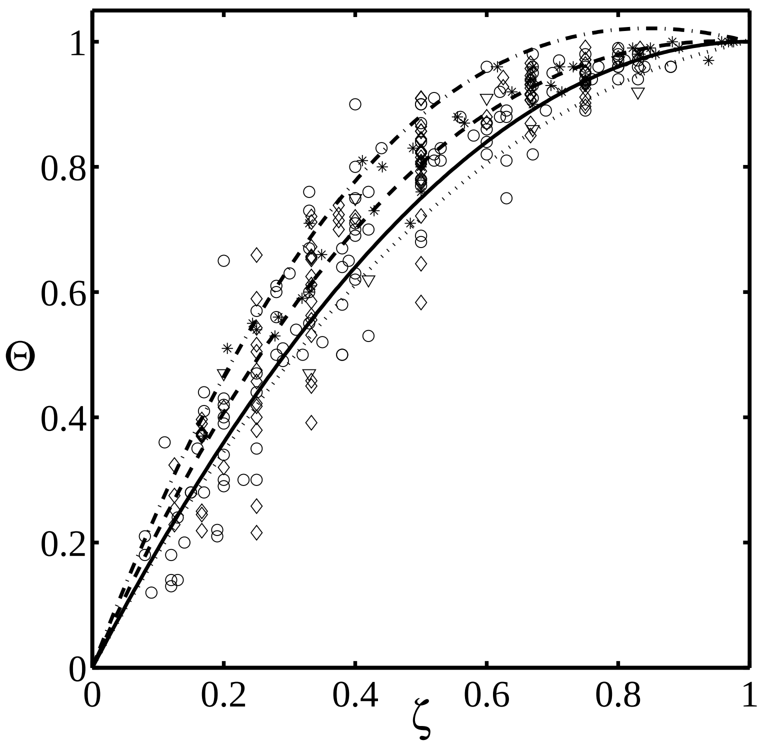

Figure 2: Dimensionless temperature $\Theta \equiv (\theta - \theta_D)/(\theta_H - \theta_D)$ as a function of dimensionless depths $\zeta \equiv (z - D)/(H - D)$. Open symbols show data from measurements in Lake Krasnoe (circles), in Lake Velen (triangles), and in Lake Mendota (diamonds). Asterisks show data from laboratory experiments. Solid curve shows a phenomenological approximation (6). Other curves show the travelling wave-type analytical solution (5) with different values of dimensionless parameters E, Π_D and Π_H: $E = 1$, $\Pi_D = 2$ and $\Pi_H = 1$, dotted curve; $E = 1.5$, $\Pi_D = 3$ and $\Pi_H = 1.5$, dashed curve; and $E = 2$, $\Pi_D = 4$ and $\Pi_H = 2$, dot-dashed curve.

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