1 Multifractal analysis

Multifractal structure appears as a result of some kind of self-similar multiplicative process. Let us imagine a line (Fig. A1) with normalized mass and length, both equal to unity. Then, at the first step, the line is divided into K equal parts S_i $(i = 1, ..., K)$. In our example (Fig. A1) $K = 2$. Then, the mass is distributed along the line in such a way, that the mass of the part with number i is M_i. The mass of the line is equal to unity, therefore $\sum_{i=1}^{K} M_i = 1$. At the next step each part S_i is also divided into K equal parts S_{il} $(l = 1, ..., K)$ and the mass is redistributed along these parts S_i in the same proportion as at the first step. Thus the mass of part S_{il} is $M_i M_l$ (l=1,...,K). After n steps the line will be divided into K^n parts $S_{i_1 i_2 ... i_n}$ with mass $M_{i_1} M_{i_2} ... M_{i_n}$ (the indexes $i_1, i_2, ..., i_n$ are varying from 1 to K).

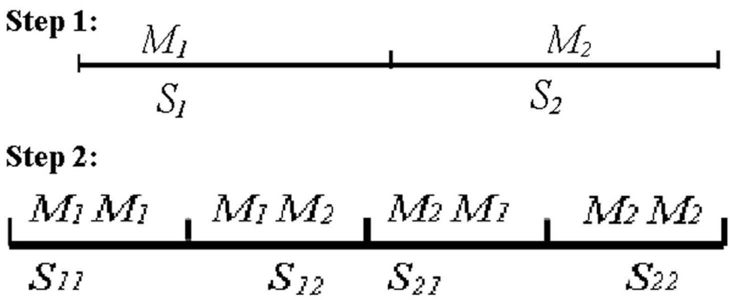

Figure A1: Multifractal structure generation process

Such multiplicative process with the parameters $M_1, M_2, ..., M_K$ generates multifractal distribution of mass along the line. Multifractal functions, generated by binomial ($K = 2$) multiplicative process with the parameters $M_1 = 0.6$ and $M_2 = 0.4$ after 10 steps are shown on Fig. A2.

The left plot on Fig. A2 corresponds to the case when in each step of multiplicative process, the smaller part of mass goes to the right side of each considered segment of line (non-mixed multiplicative process). In the case, which is shown on the right side of Fig. A2, in half of cases the smaller part of mass goes to the left side of the line segment (mixed multiplicative process). Fig. A3 demonstrate more inhomogeneous multifractal structures, which have been generated by binomial ($K = 2$) multiplicative process (non-mixed and mixed) with parameters $M_1 = 0.8, M_2 = 0.2$.

In the case of binomial multiplicative process the code $(i_1 i_2 ... i_n)$ of the part $S_j = S_{i_1 i_2 ... i_n}$ of the considered line consist of the numbers '1' and '2'. If m is the number of the values '1' in the code of the part S_j, then the mass of this part (the value of multifractal measure function for this part) can be written as follows:

$$p_j = {M_1}^m {M_2}^{n-m} = M_1^{n\varphi_1} M_2^{n\varphi_2};$$
$$\varphi_1 = \frac{m}{n};\ \varphi_2 = \frac{n-m}{n} = 1 - \varphi_1.$$

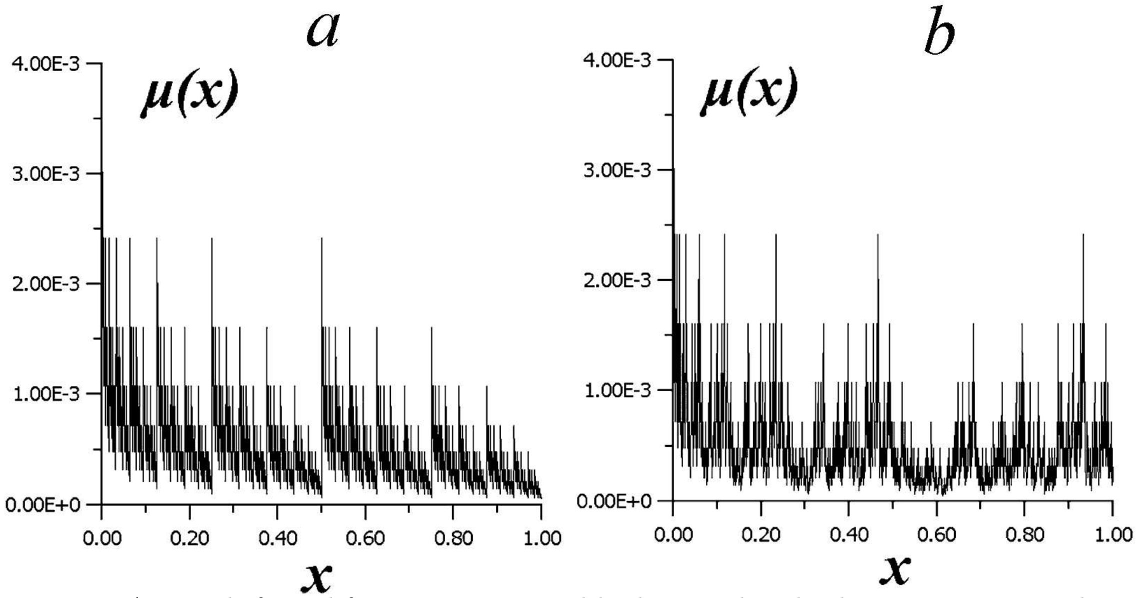

Figure A2: Multifractal function, generated by binomial multiplicative process with parameters $M_1 = 0.6$, $M_2 = 0.4$: a - non-mixed multiplicative process, b - mixed multiplicative process.

The number of parts of the line with mass $p = p_j(\varphi_1)$ can be calculated as:

$$N_n(\varphi_1) = \frac{n!}{(\varphi_1 n)!(\varphi_2 n)!},$$

using the well-known formula from combinatorics. According to Stirling formula, $n! \rightarrow \sqrt{2\pi} n^{n+1/2} e^{-n}$, when $n \rightarrow \infty$. Therefore,

$$N_n(\varphi_1) = \frac{exp(-n(\varphi_1 \ln \varphi_1 + \varphi_2 \ln \varphi_2))}{\sqrt{2\pi n \varphi_1 \varphi_2}}.$$

The size Δ of the part S_j is equal to 2^{-n}. Then, if $n \rightarrow \infty$, $N_n(\varphi_1) \sim g(\varphi_1)\Delta^{-f(\varphi_1)}$.

The Holder exponents α_j for each part S_j of the line are defined as follows:

$$\alpha_j = \lim_{\Delta \to 0} \frac{\ln(p_j)}{\ln(\Delta)}; \quad p_j = \Delta^{\alpha_j} \tag{A1}$$

Then, as $\alpha = \alpha(\varphi_1)$, $N_n(\varphi_1(\alpha)) \sim g(\varphi_1(\alpha))\Delta^{-f(\varphi_1(\alpha))}$.

Multifractal spectrum ($f(\alpha)$ curve) is defined as:

$$f(\alpha) = \lim_{\Delta \to 0} \frac{\ln(N_n(\varphi_1(\alpha))}{\ln(1/\Delta)}. \tag{A2}$$

It can be seen that $f(\alpha)$ is the fractal dimension of the subset of segments of line with Holder exponent equal to α.

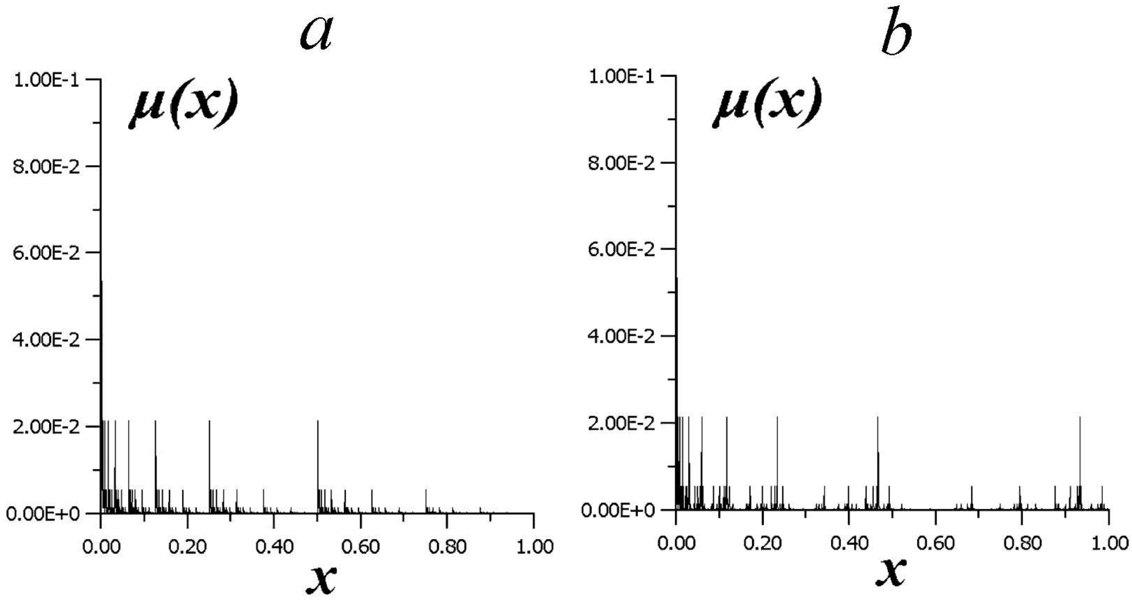

Figure A3: Multifractal function, generated by binomial multiplicative process with parameters $M_1 = 0.8$, $M_2 = 0.2$: a - non-mixed multiplicative process, b - mixed multiplicative process.

The definition of multifractal spectrum $f(\alpha)$ is one of the ways of description of multifractal structures. Another way is based on calculation of generalized fractal dimensions.

Let us introduce the Renyi entropy of order q:

$$\ln(I_q(\Delta)) = \frac{\ln(\sum_j p_j^q(\Delta))}{1-q}, \text{ if } q \neq 1$$
$$\ln(I_q(\Delta)) = -\sum_i p_i(\Delta)\ln(p_i(\Delta)), \text{ if } q = 1. \qquad \text{(A3)}$$

Then, the generalized fractal dimension of the order q is defined as:

$$d(q) = \lim_{\Delta \to 0} \frac{\ln(I_q(\Delta))}{\ln(1/\Delta)} \qquad \text{(A4)}$$

The value $d(0)$ is fractal dimension, and $d(2)$ — correlation dimension of the considered mass distribution.

The spectrums of generalized fractal dimensions $d(q)$ for multifractal structures presented on the right sides of Fig. A2 and A3, are shown on Fig. A4 by the curves 1 and 2 respectively. The difference $w = d(\infty) - d(-\infty)$ is larger for the more inhomogeneous case (Fig. A3).

It is possible to show that the Holder exponent α and $f(\alpha)$ curve can be retrieved from the spectrum of generalized fractal dimensions $d(q)$ using the transforms:

$$\alpha(q) = \frac{d}{dq}((q-1)d(q))$$
$$f(\alpha(q)) = q - d(q)(q-1) \qquad \text{(A5)}$$

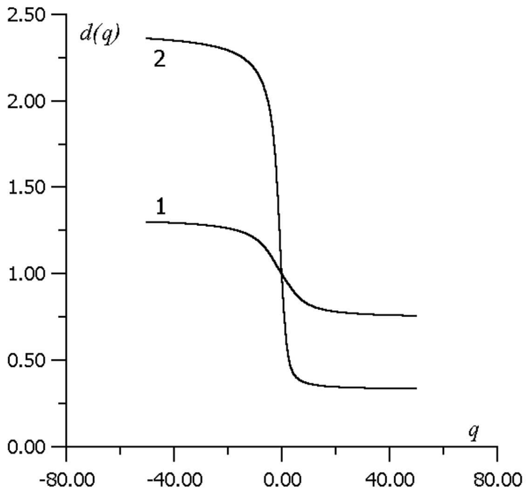

Figure A4: The spectrums of generalized fractal dimensions $d(q)$ for multifractal functions which are shown on Fig. 3b (curve 1), 4b (curve 2).

The multifractal spectrum $f(\alpha)$ contains important information about the parameters of multiplicative process underlying generation of the considered multifractal distribution function generation. The multifactal spectrums calculated for multifractal distributions, which are shown on the right side of Fig. A2 and A3, are demonstrated on Fig. A5 by the curves 1 and 2 respectively.

The maximal value of $f(\alpha)$ is equal to fractal dimension $d(0)$. The values α_{min} and α_{max} are linked correspondingly to maximal (M_{max}) and minimal (M_{min}) parameters of multiplicative process and to extremal values of spectrum of generalized fractal dimensions:

$$\alpha_{min} = d(\infty) = \frac{1}{\ln K}\ln(\frac{1}{M_{max}});$$
$$\alpha_{max} = d(-\infty) = \frac{1}{\ln K}\ln(\frac{1}{M_{min}}). \tag{A6}$$

The value α_{min} (α_{max}) describes the scaling in more densely (sparsely) populated domains of the considered region. It can be seen (Fig A5) that in the case of distribution with more high contrasts (Fig. A3) the width of multifractal spectrum is larger.

The Holder exponent $S = \alpha_1$, satisfying the equation

$$\frac{df(\alpha_1)}{d\alpha_1} = 1,$$

is equal to the generalized fractal dimension $d(1)$ and can be expressed as a function of the parameters of multiplicative process:

$$S = -\sum_{i=1}^{K} M_i \log_K M_i. \tag{A7}$$

It can be called the entropy of multifractal measure generation process since it has a form of informational entropy. The values of entropy S for the multifractal spectrums, which are calculated for distribution functions shown on Fig. A2 and A3, are correspondingly $S = 0.98$ and $S = 0.73$. It is seen that the entropy S is smaller for more inhomogeneous distribution case.

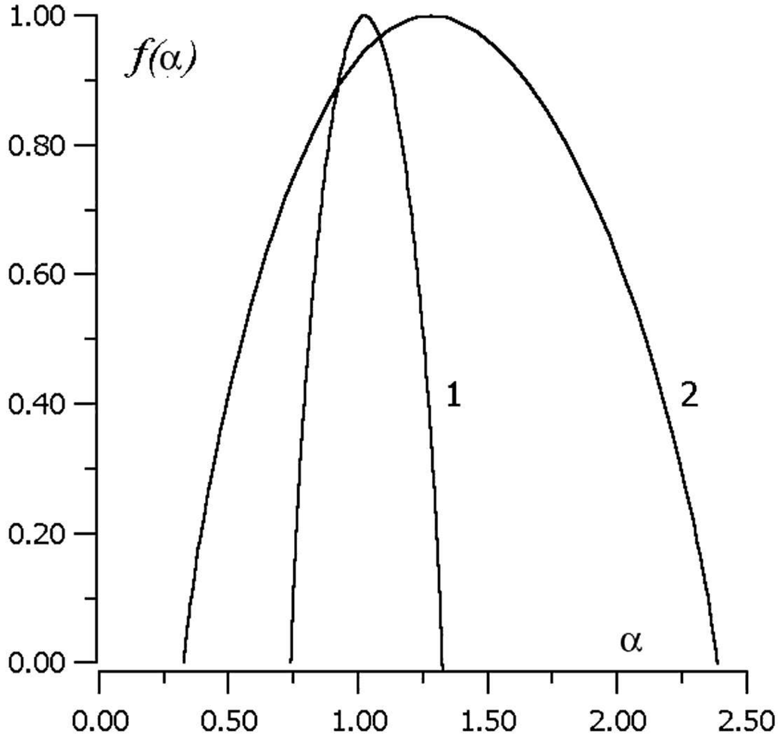

Figure A5: Multifractal spectrums $f(\alpha)$ for functions which are shown on Fig. 3b (curve 1), 4b (curve 2).